2023-24 MATH2048: Honours Linear Algebra II

Homework 8 Answer

Due: 2023-11-13 (Monday) 23:59

For the following homework questions, please give reasons in your solutions. Scan your solutions and submit it via the Blackboard system before due date.

1. Let V be an inner product space over F, show that

   (a) If $x, y \in V$ are orthogonal, then $\|x+y\|^2 = \|x\|^2 + \|y\|^2$.

   (b) $\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2$ for all $x, y \in V$ (The *parallelogram law*).

   (c) Let $v_1, v_2, \ldots, v_k$ be an orthogonal set in V, and let $a_1, a_2, \ldots, a_k \in F$. Then $\|\sum_{i=1}^k a_i v_i\|^2 = \sum_{i=1}^k |a_i|^2 \|v_i\|^2$.

   Solution.

   (a) $\|x+y\|^2 = \langle x+y, x+y\rangle = \langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle = \|x\|^2 + \|y\|^2$ since $\langle x, y\rangle = \langle y, x\rangle = 0$.

   (b)

   $$\begin{aligned}&\|x+y\|^2 + \|x-y\|^2\\ =&\langle x+y, x+y\rangle + \langle x-y, x-y\rangle\\ =&\langle x, x\rangle + \langle x, y\rangle + \langle y, x\rangle + \langle y, y\rangle + \langle x, x\rangle - \langle x, y\rangle - \langle y, x\rangle + \langle y, y\rangle\\ =&2\|x\|^2 + 2\|y\|^2\end{aligned}$$

   (c) since $\langle v_i, v_j\rangle = 0$ when $i \neq j$, one has $\|\sum_{i=1}^k a_i v_i\|^2 = \langle \sum_{i=1}^k a_i v_i, \sum_{i=1}^k a_i v_i\rangle = \sum_{i=1}^k \sum_{j=1}^k a_i \overline{a_j} \langle v_i, v_j\rangle = \sum_{i=1}^k |a_i|^2 \|v_i\|^2$

2. Prove that if V is an inner product space, then $|\langle x, y\rangle| = \|x\| \cdot \|y\|$ if and only if one of the vectors x or y is a multiple of the other. Try to derive a similar result for the equality $\|x+y\| = \|x\| + \|y\|$.

   Solution.

- ($\Leftarrow$)

  If $x = cy$ for $c \in F$. Then $|\langle x, y\rangle| = |\langle x, cx\rangle| = |c|\|x\|^2 = \|x\|\|y\|$.

- ($\Rightarrow$)

  If $y = 0$, then $y = 0x$.
  If $y \neq 0$, let $a = \dfrac{\langle x, y\rangle}{\|y\|^2}$ and $z = x - ay$. Then $\langle y, z\rangle = \langle y, x - \dfrac{\langle x, y\rangle}{\|y\|^2} y\rangle = \langle y, x\rangle - \overline{\dfrac{\langle x, y\rangle}{\|y\|^2}}\langle y, y\rangle = 0$, which implies y is orthogonal to z. Note that $|a|^2 = a \cdot \overline{a} = \dfrac{\langle x, y\rangle}{\|y\|^2} \cdot \overline{\dfrac{\langle x, y\rangle}{\|y\|^2}} = \dfrac{|\langle x, y\rangle|^2}{\|y\|^4} = \dfrac{\|x\|^2\|y\|^2}{\|y\|^4} = \dfrac{\|x\|^2}{\|y\|^2}$. So $\|x\|^2 = \|ay + z\|^2 = \|ay\|^2 + \|z\|^2 = \dfrac{\|x\|^2}{\|y\|^2}\|y\|^2 + \|z\|^2 = \|x\|^2 + \|z\|^2$ which implies $\|z\|^2 = 0 \implies z = 0$. Therefore, $x = ay + 0 = \dfrac{\langle x, y\rangle}{\|y\|^2} y$.

3. Let $V = M_{2\times 2}(\mathbb{C})$. Apply the Gram–Schmidt process to

$$S = \left\{ \begin{pmatrix} 1-i & -2-3i \\ 2+2i & 4+i \end{pmatrix}, \begin{pmatrix} 8i & 4 \\ -3-3i & -4+4i \end{pmatrix}, \begin{pmatrix} -25-38i & -2-13i \\ 12-78i & -7+24i \end{pmatrix} \right\}$$

to obtain an orthogonal basis S' for $span(S)$. Then normalize the vectors in S' to obtain an orthonormal basis S''.

Solution. Recall that the inner product of $M_{2\times 2}(\mathbb{C})$ is $\langle A, B\rangle = \operatorname{tr}(B^*A)$.

Let $S = \{w_1, w_2, w_3\}$. Construct $S' = \{v_1, v_2, v_3\}$ by

- $v_1 = w_1 = \begin{pmatrix} 1-i & -2-3i \\ 2+2i & 4+i \end{pmatrix}$.
- $v_2 = w_2 - \dfrac{\langle w_2, v_1\rangle}{\|v_1\|^2} v_1 = \begin{pmatrix} 6i & -1-1i \\ 1-3i & 1+1i \end{pmatrix}$.
- $v_3 = w_3 - \dfrac{\langle w_3, v_1\rangle}{\|v_1\|^2} v_1 - \dfrac{\langle w_3, v_2\rangle}{\|v_2\|^2} v_2 = \begin{pmatrix} -2-43i & 1-21i \\ -68i & 34i \end{pmatrix}$.

Then normalize each vectors in S' to obtains

$$S'' = \left\{ \begin{pmatrix} 1-i & -2-3i \\ 2+2i & 4+i \end{pmatrix} / \sqrt{40}, \begin{pmatrix} 6i & -1-1i \\ 1-3i & 1+1i \end{pmatrix} / \sqrt{50}, \begin{pmatrix} -2-43i & 1-21i \\ -68i & 34i \end{pmatrix} / \sqrt{8075} \right\}.$$

4. Let V be a finite-dimensional inner product space over F.

   (a) *Parseval's Identity.* Let $\{v_1, v_2, \ldots, v_n\}$ be an orthonormal basis for V. For any $x, y \in V$ prove that $\langle x, y\rangle = \sum_{i=1}^{n} \langle x, v_i\rangle \overline{\langle y, v_i\rangle}$.

(b) Use (a) to prove that if β is an orthonormal basis for V with inner product $\langle\cdot,\cdot\rangle$, then for any $x, y \in V$, we have $\langle [x]_\beta, [y]_\beta\rangle' = \langle x, y\rangle$, where $\langle\cdot,\cdot\rangle'$ is the standard inner product on F^n.

Solution.

(a) Since $\{v_1, ..., v_n\}$ is the orthonormal basis, one has $x = \sum_{i=1}^n \langle x, v_i\rangle v_i$ and $y = \sum_{i=1}^n \langle y, v_i\rangle v_i$. Then

$$\begin{aligned}\langle x, y\rangle &= \langle \sum_{i=1}^n \langle x, v_i\rangle v_i, \sum_{j=1}^n \langle y, v_j\rangle v_j\rangle \\ &= \sum_{i=1}^n \sum_{j=1}^n \langle x, v_i\rangle \overline{\langle y, v_j\rangle} \langle v_i, v_j\rangle \\ &= \sum_{k=1}^n \langle x, v_k\rangle \overline{\langle y, v_k\rangle}\end{aligned}$$

(b) Let $\beta = \{v_1, ..., v_n\}$. Then $\langle x, y\rangle = \sum_{i=1}^n \langle x, v_i\rangle \overline{\langle y, v_i\rangle} = [y]_\beta^* [x]_\beta = \langle [x]_\beta, [y]_\beta\rangle'$.

5. (a) *Bessel's Inequality.* Let V be an inner product space, and let $S = \{v_1, v_2, \ldots, v_n\}$ be an orthonormal subset of V. Prove that for any $x \in V$ we have $\|x\|^2 \geq \sum_{i=1}^n |\langle x, v_i\rangle|^2$.

(b) In the context of (a), prove that Bessel's inequality is an equality if and only if $x \in span(S)$.

Solution.

(a) $W = \text{span}(S)$ is a finite-dimensional subspace of the innver product space V. There exist unique vectors $u \in W$ and $z \in W^\perp$ such that $x = u + z$. Since S is an orthonormal basis for W, one has $u = \sum_{i=1}^n \langle x, v_i\rangle v_i$. Since u and z are orthogonal, one has $\|x\|^2 = \|u + z\|^2 = \|u\|^2 + \|z\|^2 \geq \|u\|^2 = \sum_{i=1}^n |\langle x, v_i\rangle|^2$.

(b) The equality holds iff $\|z\|^2 = 0$ iff $z = 0$ iff $x = u \in W = \text{span}(S)$.